Chap. 11B - Rigid Body Rotation

A PowerPoint Presentation by

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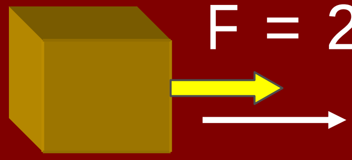
Objectives: After completing this module, you should be able to:

- Define and calculate the moment of inertia for simple systems.
- Define and apply the concepts of Newton's second law, rotational kinetic energy, rotational work, rotational power, and rotational momentum to the solution of physical problems.
- Apply principles of conservation of energy and momentum to problems involving rotation of rigid bodies.

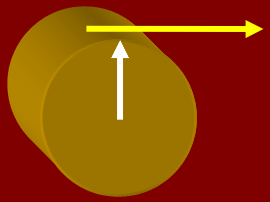
Inertia of Rotation

Consider Newton’s second law for the inertia of rotation to be patterned after the law for translation.



F = 20 N

$a = 4\text{ m/s}^2$



F = 20 N

R = 0.5 m

$\alpha = 2\text{ rad/s}^2$

Linear Inertia, *m*

$$m = \frac{24\text{ N}}{4\text{ m/s}^2} = 5\text{ kg}$$

Rotational Inertia, *I*

$$I = \frac{\tau}{\alpha} = \frac{(20\text{ N})(0.5\text{ m})}{4\text{ m/s}^2} = 2.5\text{ kg m}^2$$

Force does for translation what torque does for rotation:

Rotational Kinetic Energy

Consider tiny mass *m:*

$K = \frac{1}{2} m v^2$

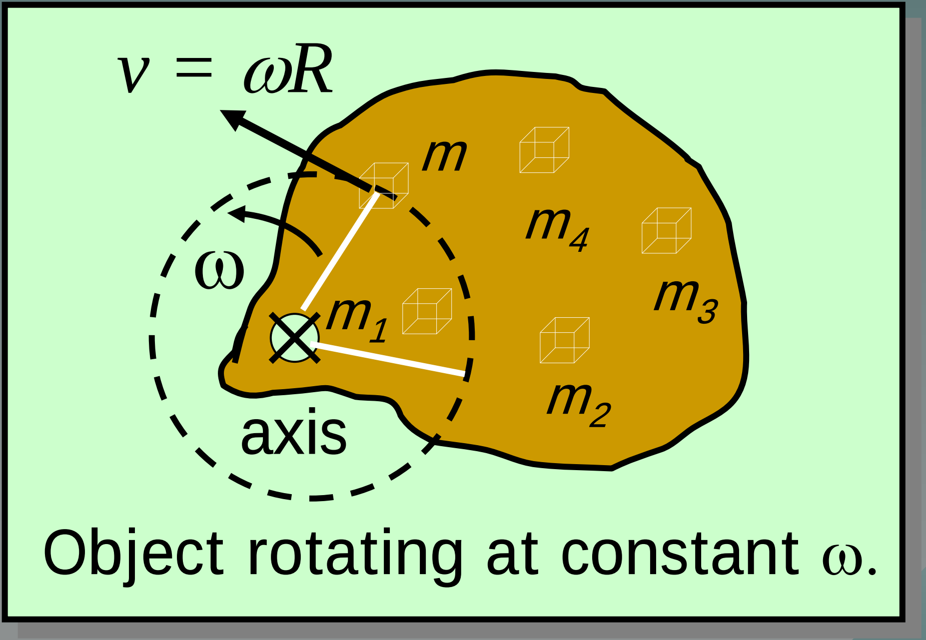
$K = \frac{1}{2} m (\omega R)^2$

$K = \frac{1}{2} (m R^2) \omega^2$

Sum to find K total:

$K = \frac{1}{2} (\Sigma m R^2) \omega^2$

($\frac{1}{2} \omega^2$ same for all *m*)



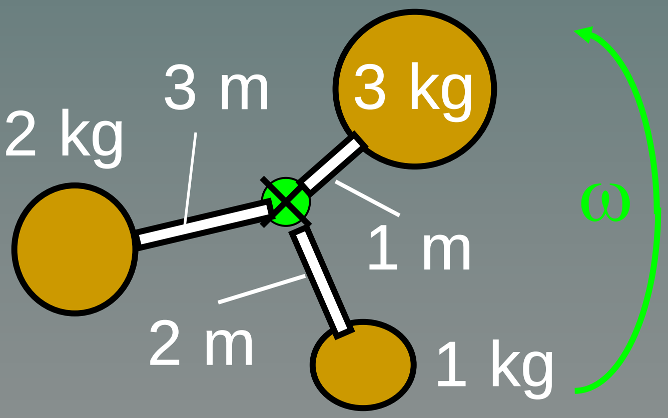
Object rotating at constant ω.

Rotational Inertia Defined:

$$\boxed{I = \Sigma m R^2}$$

Example 1: What is the rotational kinetic energy of the device shown if it rotates at a constant speed of 600 rpm?

First: $I = \Sigma mR^2$



$$I = (3\text{ kg})(1\text{ m})^2 + (2\text{ kg})(3\text{ m})^2 + (1\text{ kg})(2\text{ m})^2$$

$I = 25\text{ kg m}^2$

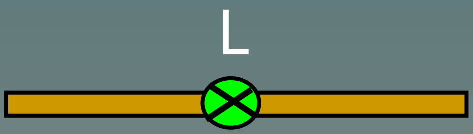
$\omega = 600\text{ rpm} = 62.8\text{ rad/s}$

$K = \frac{1}{2}Iw^2 = \frac{1}{2}(25\text{ kg m}^2)(62.8\text{ rad/s})^2$

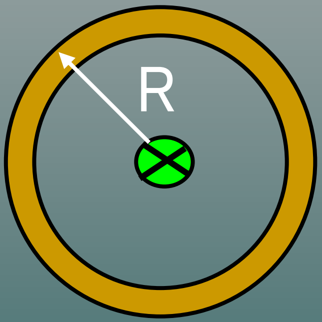
$\boxed{K = 49{,}300\text{ J}}$

Common Rotational Inertias

$$I = \tfrac{1}{3} mL^2$$

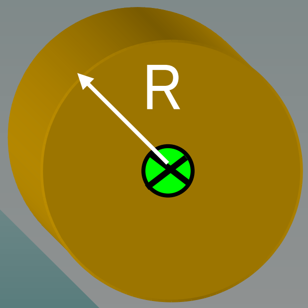


$$I = \tfrac{1}{12} mL^2$$



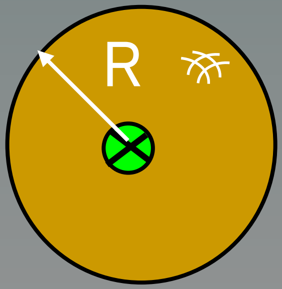
$$I = mR^2$$

Hoop



$$I = \tfrac{1}{2} mR^2$$

Disk or cylinder



$$I = \tfrac{2}{5} mR^2$$

Solid sphere

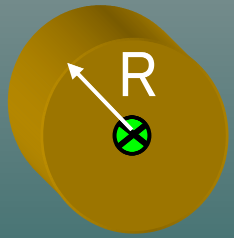
Example 2: A circular hoop and a disk each have a mass of 3 kg and a radius of 30 cm. Compare their rotational inertias.

$$I = mR^2 = (3\text{ kg})(0.2\text{ m})^2$$

$$\boxed{I = 0.120\text{ kg m}^2}$$

$$\boxed{I = mR^2}$$

Hoop



$$\boxed{I = \tfrac{1}{2}mR^2}$$

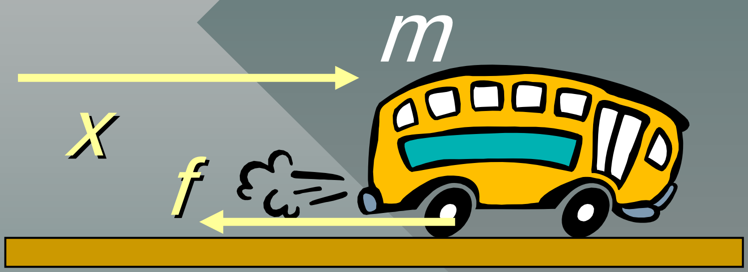
Disk

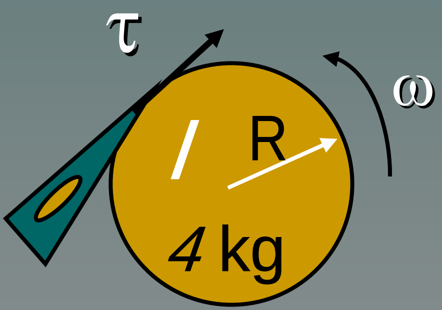
$$I = \tfrac{1}{2}mR^2 = \tfrac{1}{2}(3\text{ kg})(0.2\text{ m})^2$$

$$\boxed{I = 0.0600\text{ kg m}^2}$$

Important Analogies

For many problems involving rotation, there is an analogy to be drawn from linear motion.





$\omega_o = 50$ rad/s

$\tau = 40$ N m

A resultant force *F* produces negative acceleration *a* for a mass *m*.

$$\boxed{F = ma}$$

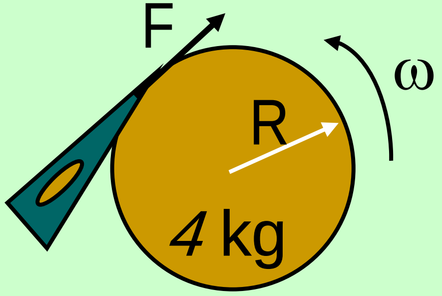
A resultant torque τ produces angular acceleration α of disk with rotational inertia *I*.

$$\boxed{\tau = I\alpha}$$

Newton's 2nd Law for Rotation

How many revolutions required to stop?

$$\boxed{\tau = I\alpha}$$



$\omega_o = 50$ rad/s

R = 0.20 m

F = 40 N

$$F\cancel{R} = (\tfrac{1}{2}mR^{\cancel{2}})\alpha$$

$$\alpha = \frac{2F}{mR} = \frac{2(40\text{N})}{(4\text{ kg})(0.2\text{ m})}$$

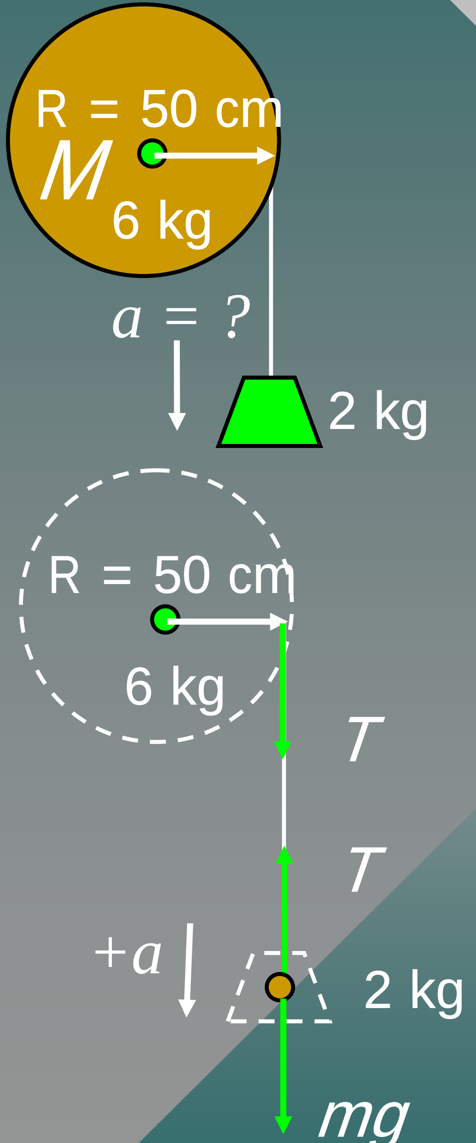
$$\boxed{\alpha = 100\text{ rad/s}^2}$$

$$2\alpha\theta = \cancelto{0}{\omega_f^2} - \omega_o^2$$

$$\theta = \frac{-\omega_0^2}{2\alpha} = \frac{-(50\text{ rad/s})^2}{2(100\text{ rad/s}^2)}$$

$$\boxed{\theta = 12.5\text{ rad} = 1.99\text{ rev}}$$

Example 3: What is the linear acceleration of the falling 2-kg mass?



Apply Newton's 2nd law to rotating disk:

$$\tau = I\alpha \quad \Rightarrow \quad T\cancel{R} = (\tfrac{1}{2}M\cancel{R}^2)\alpha$$

$$T = \tfrac{1}{2}MR\alpha \quad \text{but} \quad a = \alpha R; \quad \alpha = \frac{a}{R}$$

$$T = \tfrac{1}{2}M\cancel{R}\left(\frac{a}{\cancel{R}}\right); \quad \text{and} \quad \boldsymbol{T = \tfrac{1}{2}Ma}$$

Apply Newton's 2nd law to falling mass:

$$mg - T = ma \qquad mg - \tfrac{1}{2}Ma = ma$$

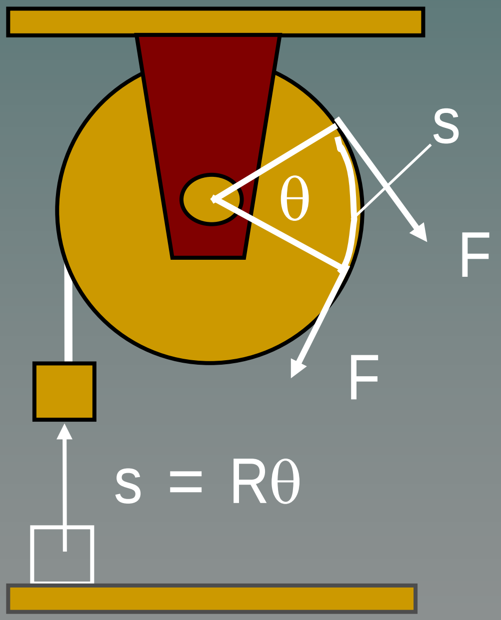
$$(2\text{ kg})(9.8\text{ m/s}^2) - \tfrac{1}{2}(6\text{ kg})\,\boldsymbol{a} = (2\text{ kg})\,\boldsymbol{a}$$

$$19.6\text{ N} - (3\text{ kg})\,\boldsymbol{a} = (2\text{ kg})\,\boldsymbol{a}$$

$$\boxed{a = 3.92\text{ m/s}^2}$$

Work and Power for Rotation

$$\text{Work} = Fs = FR\theta \qquad \tau = FR$$



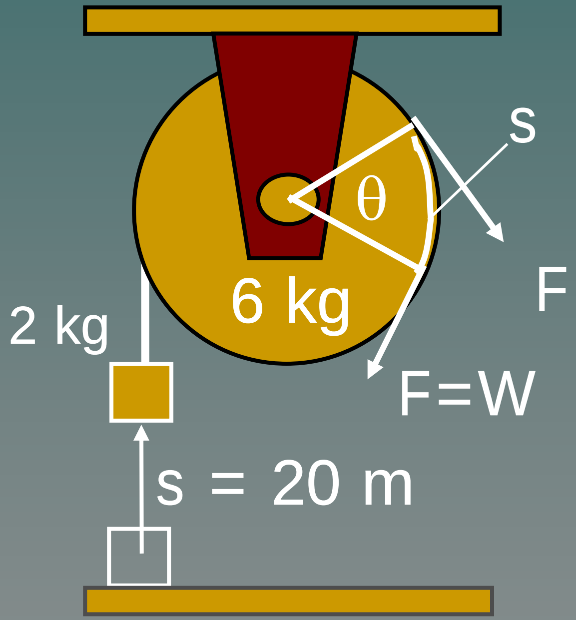
$$\boxed{\text{Work} = \tau\theta}$$

$$\text{Power} = \frac{\text{Work}}{t} = \frac{\tau\theta}{t} \qquad \bar{\omega} = \frac{\theta}{t}$$

$$\boxed{\text{Power} = \tau\,\bar{\omega}}$$

Power = Torque x average angular velocity

Example 4: The rotating disk has a radius of 40 cm and a mass of 6 kg. Find the work and power if the 2-kg mass is lifted 20 m in 4 s.



$$\text{Work} = \tau\theta = FR\,\theta$$

$$\theta = \frac{s}{R} = \frac{20\text{ m}}{0.4\text{ m}} = 50\text{ rad}$$

$$F = mg = (2\text{ kg})(9.8\text{ m/s}^2);\quad F = 19.6\text{ N}$$

$$\text{Work} = (19.6\text{ N})(0.4\text{ m})(50\text{ rad})$$

$$\boxed{\text{Work} = 392\text{ J}}$$

$$\text{Power} = \frac{\text{Work}}{t} = \frac{392\text{ J}}{4\text{s}}$$

$$\boxed{\text{Power} = 98\text{ W}}$$

The Work-Energy Theorem

Recall for linear motion that the work done is equal to the change in linear kinetic energy:

$$\boxed{Fx = ½mv_f^2 - ½mv_0^2}$$

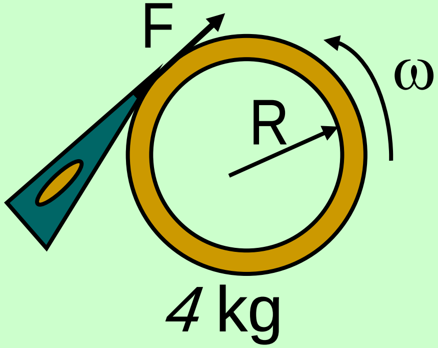
Using angular analogies, we find the rotational work is equal to the change in rotational kinetic energy:

$$\boxed{\tau\theta = ½I\omega_f^2 - ½I\omega_0^2}$$

Applying the Work-Energy Theorem:

What work is needed to stop wheel rotating:

$$\boxed{Work = \Delta K_r}$$



$\omega_o = 60$ rad/s

R = 0.30 m

F = 40 N

First find *I* for wheel: $I = mR^2$ = (4 kg)(0.3 m)^2 = 0.36 kg m^2

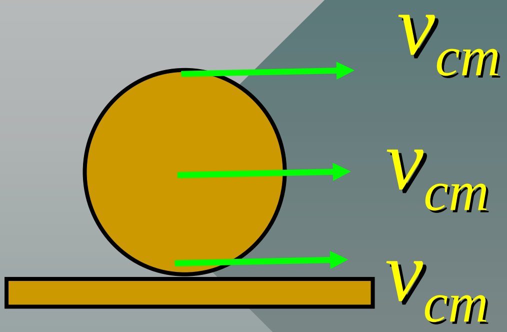
$$\tau\theta = \cancelto{0}{\tfrac{1}{2}I\omega_f^2} - \tfrac{1}{2}I\omega_0^2$$

$$Work = -\tfrac{1}{2}I\omega_o^2$$

Work = -½(0.36 kg m^2)(60 rad/s)^2

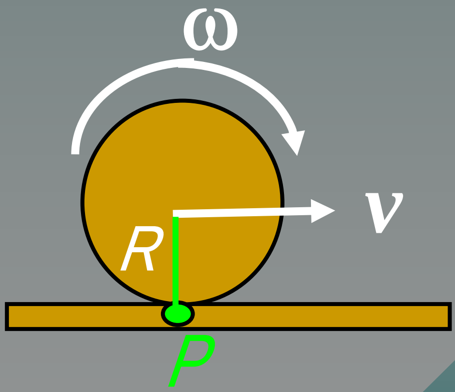
$$\boxed{\text{Work} = -648\ \text{J}}$$

Combined Rotation and Translation



First consider a disk sliding without friction. The velocity of any part is equal to velocity v_{cm} of the center of mass.

Now consider a ball rolling without slipping. The angular velocity ω about the point P is same as ω for disk, so that we write:



$$\boxed{\omega = \frac{v}{R}}$$

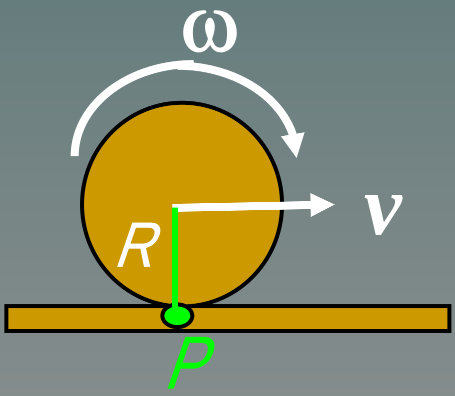
Or

$$\boxed{v = \omega R}$$

Two Kinds of Kinetic Energy

Kinetic Energy of Translation: $K = \frac{1}{2}mv^2$

Kinetic Energy of Rotation: $K = \frac{1}{2}I\omega^2$



Total Kinetic Energy of a Rolling Object:

$$\boxed{K_T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2}$$

Angular/Linear Conversions

In many applications, you must solve an equation with both angular and linear parameters. It is necessary to remember the bridges:

Displacement:	$s = \theta R$ $\quad\quad$ $\theta = \frac{s}{R}$
Velocity:	$v = \omega R$ $\quad\quad$ $\omega = \frac{v}{R}$
Acceleration:	$v = \alpha R$ $\quad\quad$ $a = \frac{\alpha}{R}$

Translation or Rotation?

If you are to solve for a linear parameter, you must convert all angular terms to linear terms:

$$\theta = \frac{s}{R} \qquad \omega = \frac{v}{R} \qquad a = \frac{\alpha}{R} \qquad I = (?)mR^2$$

If you are to solve for an angular parameter, you must convert all linear terms to angular terms:

$$s = \theta R \qquad v = \omega R \qquad v = \alpha R$$

Example (a): Find velocity *v* of a disk if given its total kinetic energy *E*.

$$\boxed{\text{Total energy: } E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2}$$

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2; \quad I = \tfrac{1}{2}mR^2; \quad \omega = \frac{v}{R}$$

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m\cancel{R^2}\right)\left(\frac{v^2}{\cancel{R^2}}\right); \qquad E = \tfrac{1}{2}mv^2 + \tfrac{1}{4}mv^2$$

$$E = \frac{3mv^2}{4} \qquad \text{or} \qquad v = \sqrt{\frac{4E}{3m}}$$

Example (b) Find angular velocity ω of a disk given its total kinetic energy *E*.

Total energy: $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2; \quad I = \tfrac{1}{2}mR^2; \quad v = \omega R$$

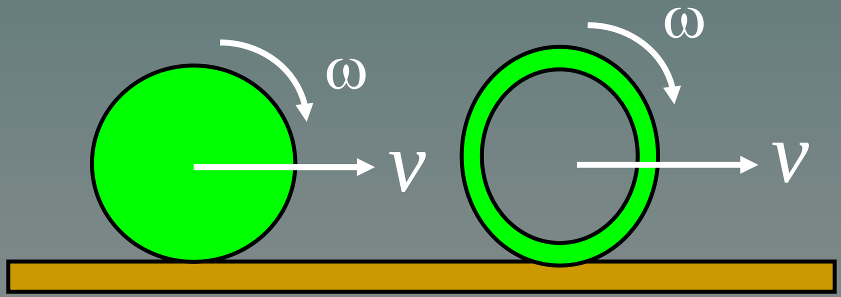
$$E = \tfrac{1}{2}m(\omega R)^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mR^2\right)\omega^2; \; E = \tfrac{1}{2}mR^2\omega^2 + \tfrac{1}{4}mR^2\omega^2$$

$$E = \frac{3mR^2\omega^2}{4} \qquad \text{or} \qquad \omega = \sqrt{\frac{4E}{3mR^2}}$$

Strategy for Problems

- Draw and label a sketch of the problem.
- List givens and state what is to be found.
- Write formulas for finding the moments of inertia for each body that is in rotation.
- Recall concepts involved (power, energy, work, conservation, etc.) and write an equation involving the unknown quantity.
- Solve for the unknown quantity.

Example 5: A circular hoop and a circular disk, each of the same mass and radius, roll at a linear speed *v*. Compare the kinetic energies.



Two kinds of energy:

$K_T = \frac{1}{2}mv^2 \quad K_r = \frac{1}{2}I\omega^2$

$$\boxed{\text{Total energy: } E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2}$$

$$\boxed{\omega = \frac{v}{R}}$$

Disk: $E = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}m\cancel{R^2}\right)\left(\frac{v^2}{\cancel{R^2}}\right)$

$$\boxed{E = \tfrac{3}{4}mv^2}$$

Hoop: $E = \frac{1}{2}mv^2 + \frac{1}{2}\left(m\cancel{R^2}\right)\left(\frac{v^2}{\cancel{R^2}}\right)$

$$\boxed{E = mv^2}$$

Conservation of Energy

The total energy is still conserved for systems in rotation and translation.

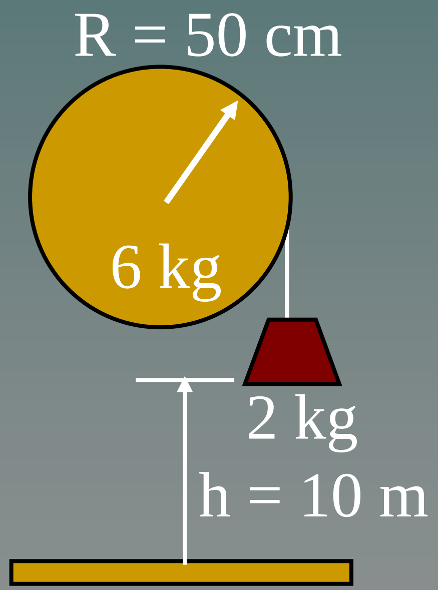
However, rotation must now be considered.

Begin: $(U + K_t + K_R)_o$ = End: $(U + K_t + K_R)_f$

Height?	mgh_o	=	mgh_f	Height?
Rotation?	$\frac{1}{2}I\omega_o^2$		$\frac{1}{2}I\omega_f^2$	Rotation?
velocity?	$\frac{1}{2}mv_o^2$		$\frac{1}{2}mv_f^2$	velocity?

Example 6: Find the velocity of the 2-kg mass just before it strikes the floor.

$$mgh_o + \cancel{\tfrac{1}{2}I\omega_o^2} + \cancel{\tfrac{1}{2}mv_o^2} = \cancel{mgh_f} + \tfrac{1}{2}I\omega_f^2 + \tfrac{1}{2}mv_f^2$$



$$mgh_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \qquad I = \tfrac{1}{2}MR^2$$

$$mgh_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}(\tfrac{1}{2}M\cancel{R^2})\left(\frac{v^2}{\cancel{R^2}}\right)$$

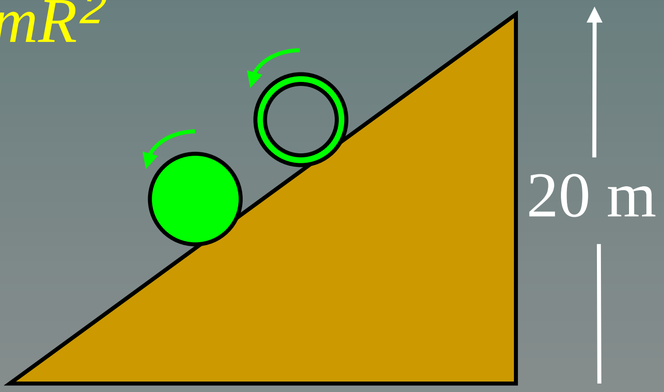
$$(2)(9.8)(10) = \tfrac{1}{2}(2)v^2 + \tfrac{1}{4}(6)v^2$$

$$2.5v^2 = 196\ \text{m}^2/\text{s}^2$$

$$\boxed{v = 8.85\ \text{m/s}}$$

Example 7: A hoop and a disk roll from the top of an incline. What are their speeds at the bottom if the initial height is 20 m?

$mgh_o = ½mv^2 + ½I\omega^2$ *Hoop:* $I = mR^2$



$$mgh_0 = ½mv^2 + ½(m\cancel{R^2})\left(\frac{v^2}{\cancel{R^2}}\right)$$

$mgh_o = ½mv^2 + ½mv^2$; $\cancel{m}gh_o = \cancel{m}v^2$

$$v = \sqrt{gh_0} = \sqrt{(9.8\text{ m/s}^2)(20\text{ m})}$$

Hoop: $\boxed{v = 14\text{ m/s}}$

Disk: $I = ½mR^2$; $mgh_o = ½mv^2 + ½I\omega^2$

$$\cancel{m}gh_0 = ½\cancel{m}v^2 + ½(½\cancel{m}\cancel{R^2})\left(\frac{v^2}{\cancel{R^2}}\right)$$

$$v = \sqrt{\tfrac{4}{3}gh_0}$$

$\boxed{v = 16.2\text{ m/s}}$

Angular Momentum Defined

Consider a particle *m* moving with velocity *v* in a circle of radius *r*.

Define angular momentum L:

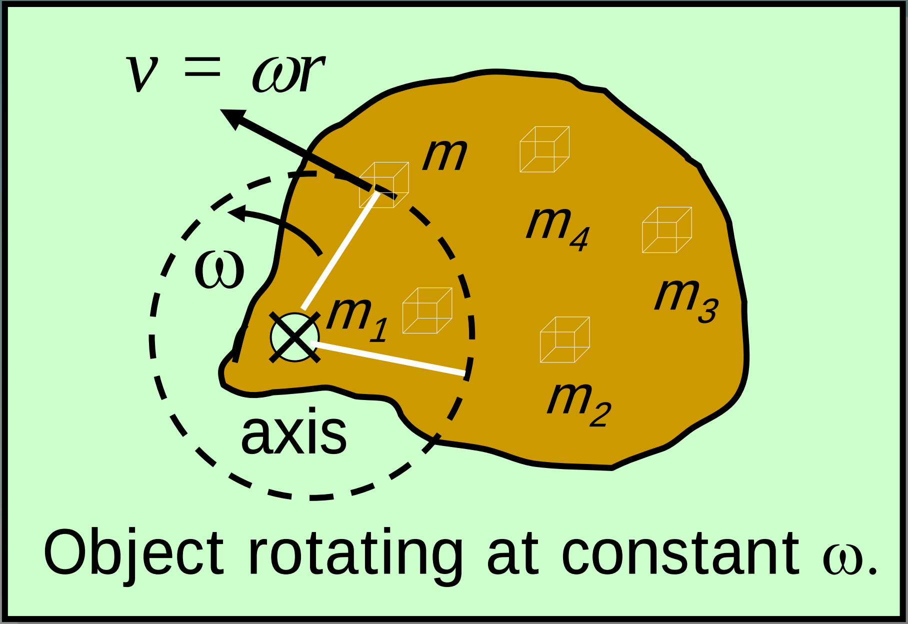
$$\boxed{L = mvr}$$

Substituting $v = \omega r$, gives:

$$L = m(\omega r)\, r = mr^2\omega$$

For extended rotating body:

$$L = (\Sigma mr^2)\, \omega$$



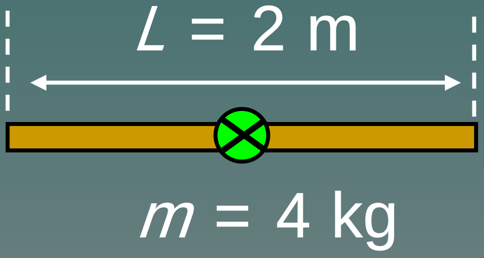
Object rotating at constant ω.

Since $I = \Sigma mr^2$, we have:

$$\boxed{L = I\omega}$$

Angular Momentum

Example 8: Find the angular momentum of a thin 4-kg rod of length 2 m if it rotates about its midpoint at a speed of 300 rpm.



For rod: $I = \frac{1}{12} mL^2 = \frac{1}{12}(4 \text{ kg})(2 \text{ m})^2$

$$\boxed{I = 1.33 \text{ kg m}^2}$$

$$\omega = \left(300 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.4 \text{ rad/s}$$

$$L = I\omega = (1.33 \text{ kg m}^2)(31.4 \text{ rad/s})^2$$

$$\boxed{L = 1315 \text{ kg m}^2\text{/s}}$$

Impulse and Momentum

Recall for linear motion the linear impulse is equal to the change in linear momentum:

$$F\,\Delta t = mv_f - mv_0$$

Using angular analogies, we find angular impulse to be equal to the change in angular momentum:

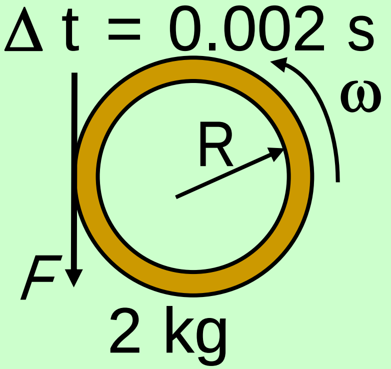
$$\tau\,\Delta t = I\omega_f - I\omega_0$$

Example 9: A sharp force of 200 N is applied to the edge of a wheel free to rotate. The force acts for 0.002 s. What is the final angular velocity?

$$I = mR^2 = (2\text{ kg})(0.4\text{ m})^2$$

$$I = 0.32\text{ kg m}^2$$

Applied torque $\tau = FR$



$\Delta t = 0.002$ s

$\omega_o = 0$ rad/s

$R = 0.40$ m

$F = 200$ N

Impulse = change in angular momentum

$$\tau\,\Delta t = I\omega_f - \cancelto{0}{I\omega_o} \Longrightarrow FR\,\Delta t = I\omega_f$$

$$\omega_f = \frac{FR\Delta t}{I} = \frac{(200\text{ N})(0.4\text{ m})(0.002\text{ s})}{0.32\text{ m}^2}$$

$$\boxed{\omega_f = 0.5\text{ rad/s}}$$

Conservation of Momentum

In the absence of external torque the rotational momentum of a system is conserved (constant).

$$I_f\omega_f - I_o\omega_o = \cancelto{0}{\tau}\,\Delta t$$

$$\boxed{I_f\omega_f = I_o\omega_o}$$

I_o = 2 kg m^2; ω_o = 600 rpm

I_f = 6 kg m^2; ω_o = ?

$$\omega_f = \frac{I_0\omega_0}{I_f} = \frac{(2\ \text{kg}\cdot\text{m}^2)(600\ \text{rpm})}{6\ \text{kg}\cdot\text{m}^2}$$

$$\boxed{\omega_f = 200\ \text{rpm}}$$

Summary – Rotational Analogies

Quantity	Linear	Rotational
Displacement	Displacement x	Radians θ
Inertia	Mass (kg)	I (kg·m^2)
Force	Newtons N	Torque N·m
Velocity	$v \longrightarrow$ “m/s ”	$\omega \longrightarrow$ Rad/s
Acceleration	$a \longrightarrow$ “m/s^2 ”	$\alpha \longrightarrow$ Rad/s^2
Momentum	mv (kg m/s)	$I\omega$ (kg·m^2·rad/s)

Analogous Formulas

Linear Motion	Rotational Motion
$F = ma$	$\tau = I\alpha$
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
$Work = Fx$	$Work = \tau\theta$
$Power = Fv$	$Power = I\omega$
$Fx = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2$	$\tau\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_o^2$

Summary of Formulas:

I = ΣmR^2

$$K = \tfrac{1}{2} I\omega^2$$

$$Work = \tau\theta$$

$$I_o\omega_o = I_f\omega_f$$

$$\tau\theta = ½I\omega_f^2 - ½I\omega_0^2$$

$$Power = \frac{\tau\theta}{t} = \tau\omega$$

Height?

Rotation?

velocity?

mgh_o

$½I\omega_o{}^2$

$½mv_o{}^2$

=

mgh_f

$½I\omega_f^2$

$½mv_f^2$

Height?

Rotation?

velocity?

CONCLUSION: Chapter 11B
Rigid Body Rotation